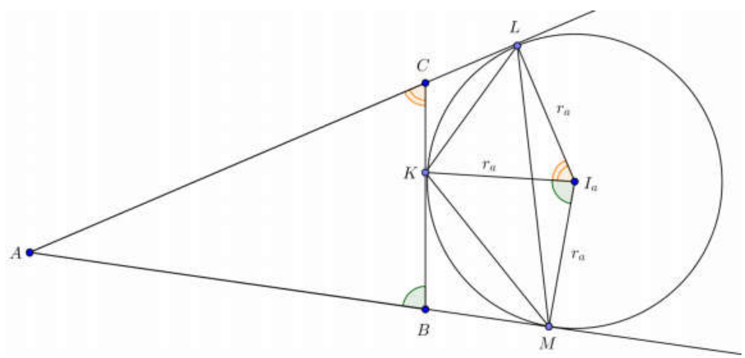
Hence
$$r \le \frac{AD \cdot AE}{AD + AE}$$
and the result follows immediately. The equality holds if and only if $\angle A$ is a right angle.

3808. *Proposed by Mehmet Şahin.*

Let ABC be a triangle with area Δ; circumradius R; exradii r_a, r_b, r_c; and excenters I_a, I_b, I_c. The excircle with centre I_a touches the sides of ABC at K, L, and M. Let Δ_1 represent the area of triangle KLM and let Δ_2 and Δ_3 be similarly defined. Prove that
$$\frac{\Delta_1 + \Delta_2 + \Delta_3}{\Delta} = \frac{r_a + r_b + r_c}{2R}.$$

Solved by A. Alt; M. Amengual Covas; Š. Arslanagić; M. Bataille; P. De; O. Geupel; J. Heuver; O. Kouba; S. Malikić; C.R. Pranesachar; C. Sánchez-Rubio; G. Tsapakidis; D. Văcaru; P. Y. Woo; T. Zvonaru; and the proposer. We present a composite of similar solutions by Arkady Alt, Miguel Amengual Covas, and Oliver Geupel.



We use the common notation
$$a = BC, \quad b = CA, \quad c = AB, \quad 2s = a + b + c.$$

Since quadrilaterals MI_aKB, I_aLCK, and MI_aLA are cyclic, we have
$$\angle MI_aK = \angle B, \quad \angle KI_aL = \angle C, \quad \text{and} \quad \angle MI_aL = \angle 180^\circ - \angle A.$$

It follows that
$$\begin{aligned}\Delta_1 &= [I_aKM] + [I_aLK] - [I_aLM] \\ &= \frac{r_a^2}{2}(\sin B + \sin C - \sin(180^\circ - A)) \\ &= \frac{r_a^2}{2}(\sin B + \sin C - \sin A) \\ &= \frac{r_a^2}{2} \cdot \frac{b + c - a}{2R} = \frac{r_a}{2R} \cdot r_a(s - a) = \frac{r_a}{2R}\Delta.\end{aligned}$$

Analogously,

$$\Delta_2 = \frac{r_b}{2R}\Delta, \quad \Delta_3 = \frac{r_c}{2R}\Delta,$$

hence the result.

3809. *Proposed by Michel Bataille.*

For positive real numbers x, y, let

$$G(x,y) = \sqrt{xy}, \quad A(x,y) = \frac{x+y}{2}, \quad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}\,.$$

Prove that

$$G(x^x, y^y) \geq (Q(x,y))^{A(x,y)}\,.$$

Solved by AN-anduud Problem Solving Group ; R. Boukharfane ; C. Curtis ; P. Deiermann and H. Wang ; O. Kouba ; K. W. Lau ; P. Perfetti ; D. Smith ; and the proposer. One incorrect solution was received. We present the solution by Paolo Perfetti.

The given inequality is equivalent to

$$x^{\frac{x}{2}}x^{\frac{y}{2}} \geq \left(\sqrt{\frac{x^2+y^2}{2}}\right)^{\frac{x+y}{2}} \iff x^{\frac{2x}{x+y}}y^{\frac{2y}{x+y}} \geq \frac{x^2+y^2}{2},$$

which upon being divided by x^2 becomes

$$\frac{y^{\frac{2y}{x+y}}}{x^{\frac{2y}{x+y}}} \geq \frac{1}{2}\left(1+\left(\frac{y}{x}\right)^2\right). \tag{1}$$

Without loss of generality, we assume that $x \leq y$. Let $t = \frac{y}{x}$. Then $t \geq 1$, $\frac{2y}{x+y} = \frac{2t}{1+t}$ and (1) becomes

$$t^{\frac{2t}{1+t}} \geq \frac{1+t^2}{2} \iff \frac{2t}{1+t}\ln t \geq \ln\left(\frac{1+t^2}{2}\right). \tag{2}$$

To prove (2), let $f(t) = \frac{2t}{1+t}\ln t - \ln\frac{1+t^2}{2}$, $t \geq 1$. Then by routine calculations, we find :

$$f'(t) = 2\left(\frac{1-t^2+(1+t^2)\ln t}{(1+t)^2(1+t^2)}\right).$$

We claim that

$$1-t^2+(1+t^2)\ln t \geq 0 \quad \text{for all} \quad t \geq 1. \tag{3}$$

Let $h(t) = \ln t - \dfrac{t^2-1}{1+t^2} = \ln t - 1 + \dfrac{2}{1+t^2}$. Then

$$h'(t) = \frac{1}{t} - \frac{4t}{(1+t^2)^2} = \frac{(1-t^2)^2}{t(1+t^2)^2} \geq 0,$$